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On the relative efficiency of some ratio and product estimators

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ABSTRACT

In this paper, the relative efficiency of (i) means per unit estimator, (ii) classical ratio and product estimators, (iii) Srivenkataramana and Srinath alternative ratio and product estimators(1976) and (iv) newly proposed alternative ratio and product estimators were examined and the conditions under which each of these estimators are preferred over the others were found and tested.

INTRODUCTION

Let N and n be the population and sample sizes respectively while $\overline{X}$ and $\overline{Y}$ be the population means for the auxiliary variable (X) and variable of interest (Y) and $\bar{x}$ and $\bar{y}$ be the means of the samples drawn from the population, $\bar{x}^*$ and $\bar{y}^*$ be the means of samples yet to be drawn from the population(Srivenkataramana and Srinath (1976)), that is the means corresponding to the (N – n) population units and $\hat{\rho}_{xy}$ be the sample coefficient of correlation between x and y.

MATERIALS AND METHODS

Determination of the mean square error and bias of the classical ratio and product estimators

Traditionally (Cochran, W.G. (1977)), we know that:-

$$mse(\bar{y}) = \frac{N-n}{Nn}\overline{Y}^2(\frac{s^2{}_y}{\overline{Y}^2}) = \overline{Y}^2(\frac{N-n}{Nn})c^2v(y)$$

Where

$$c^2v(y) = \frac{s^2{}_y}{\overline{Y}^2}$$

Let

$$\bar{y}_r = \frac{\bar{y}}{\bar{x}}\overline{X}, \text{ when } \hat{\rho}_{xy} > 0 \text{ and } \bar{y}_p = \frac{\bar{y}\bar{x}}{\overline{X}},$$

when $\hat{\rho}_{xy} < 0$,

then

$$bias(\bar{y}_r) = E(\bar{y}_r - \overline{Y}) = \overline{Y}(\frac{N-n}{Nn})[c^2v(x) - \hat{\rho}_{xy}cv(x)cv(y)]$$

$$mse(\bar{y}_r) = E(\bar{y}_r - \overline{Y})^2 = \overline{Y}^2(\frac{N-n}{Nn})[c^2v(y) - \hat{\rho}_y cv(x)cv(y) + c^2v(x)]$$

$$bias(\bar{y}_P) = E(\bar{y}_P - \overline{Y}) = \overline{Y}(\frac{N-n}{Nn})[c^2v(x) + \hat{\rho}_{xy}cv(x)cv(y)]$$

$$mse(\bar{y}_P) = E(\bar{y}_P - \overline{Y})^2 = \overline{Y}^2(\frac{N-n}{Nn})[c^2v(y) + \hat{\rho}_y cv(x)cv(y) + c^2v(x)]$$

Srivenkataramana and srinath alternative ratio and product estimators

Srivenkataramana and Srinath (1976) defined their alternative ratio and product estimators as:

$$\bar{y}_{ar1} = \frac{\bar{y}}{\overline{X}}\bar{x}^* \text{ when } \hat{\rho}_{xy} > 0, \bar{y}_{ap1} = \frac{\bar{y}\overline{X}}{\bar{x}^*} \text{ when } \hat{\rho}_{xy} < 0,$$

where $\overline{X} = f\bar{x} + (1-f)\bar{x}^*, \bar{x}^* = \overline{X}(1-(\frac{f}{1-f})\Delta_{\bar{x}})$ (1)

Conventionally, $\bar{x} = \overline{X}(1+\Delta_{\bar{x}}),$ substituting for this in (1) then

$$\bar{x}^* = \overline{X}(1-(\frac{f}{1-f})\Delta_{\bar{x}}) = \overline{X}(1-(\frac{n}{N-n})\Delta_{\bar{x}}) \quad (2)$$

where $\Delta_{\bar{x}} = \frac{\bar{x}-\overline{X}}{\overline{X}}$ therefore,

$$bias(\bar{y}_{ar1}) = E(\bar{y}_{ar1} - \overline{Y}) = -\overline{Y}(\frac{N-n}{Nn})\hat{\rho}_{xy}(\frac{n}{N-n})cv(x)cv(y)]$$

$$mse(\bar{y}_{ar1}) = E(\bar{y}_{ar1} - \overline{Y})^2 = \overline{Y}^2(\frac{N-n}{Nn})[c^2v(y) - 2\hat{\rho}_y(\frac{n}{N-n})cv(x)cv(y) + (\frac{n}{N-n})^2c^2v(x)]$$

$$bias(\bar{y}_{aP1}) = E(\bar{y}_{aP1} - \overline{Y}) = \overline{Y}(\frac{N-n}{Nn})\hat{\rho}_{xy}(\frac{n}{N-n})cv(x)cv(y)]$$

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$$mse(\bar{y}_{aP1}) = E(\bar{y}_{aP1}-\bar{Y})^2 = \bar{Y}^2(\frac{N-n}{Nn})[c^2v(y)+\widehat{\rho}_y(\frac{n}{N-n})cv(x)cv(y)+(\frac{n}{N-n})^2c^2v(x)]$$

Our alternative ratio and product estimators

Just like Srivenkataramana and Srinath (1976) alternative ratio and product

estimators, $\bar{y}_{ar1}$ and $\bar{y}_{ap1}$,let

$$\bar{y}_{ar2} = \frac{\bar{y}}{\bar{X}}\bar{x}^*, \widehat{\rho}_{xy} > 0, and, \bar{y}_{ap2} = \frac{\bar{y}\bar{X}}{\bar{x}^*}, \widehat{\rho}_{xy} < 0$$

be our alternative ratio and product estimators where

$$\bar{X} = \theta f\bar{x} + (1-\theta)\bar{x}^* \qquad (3)$$

and $\theta = \frac{1}{n} - \frac{1}{N} = \frac{N-n}{Nn}$ Substituting for $\bar{x} = \bar{X}(1+\Delta_{\bar{x}})$,then

$$\bar{x}^* = \bar{X}(1-(\frac{\theta}{1-\theta})\Delta_{\bar{x}}) = \bar{X}(1-(\frac{N-n}{Nn-N+n})\Delta_{\bar{x}}). \qquad (4)$$

Therefore,

$$bias(\bar{y}_{ar2}) = E(\bar{y}_{ar2}-\bar{Y}) = -\bar{Y}(\frac{N-n}{Nn})\widehat{\rho}_{xy}(\frac{N-n}{Nn-N+n})cv(x)cv(y)]$$

$$mse(\bar{y}_{ar2}) = E(\bar{y}_{ar2}-\bar{Y})^2$$

$$=\bar{Y}^2(\frac{N-n}{Nn})[c^2v(y)-2\widehat{\rho}_{xy}(\frac{N-n}{Nn-N+n})cv(x)cv(y)+(\frac{N-n}{Nn-N+n})^2c^2v(x)]$$

$$bias(\bar{y}_{aP2}) = E(\bar{y}_{aP2}-\bar{Y}) = \bar{Y}(\frac{N-n}{Nn})\widehat{\rho}_{xy}(\frac{N-n}{Nn-N+n})cv(x)cv(y)]$$

$$mse(\bar{y}_{aP2}) = E(\bar{y}_{aP2}-\bar{Y})^2$$

$$=\bar{Y}^2(\frac{N-n}{Nn})[c^2v(y)+\rho_y(\frac{N-n}{Nn-N+n})cv(x)cv(y)+(\frac{N-n}{Nn-N+n})^2c^2v(x)]$$

Here, it needs be pointed out that while Srivenkaramana and Srinath defined

$$\bar{x}^* = \bar{X}(1-(\frac{f}{1-f})\Delta_{\bar{x}}) = \bar{X}(1-(\frac{n}{N-n})\Delta_{\bar{x}}),$$

we defined ours to be

$$\bar{x}^* = \bar{X}(1-(\frac{\theta}{1-\theta})\Delta_{\bar{x}}) = \bar{X}(1-(\frac{N-n}{Nn-N+n})\Delta_{\bar{x}}),$$

where

$$f = \frac{n}{N} \quad \text{and} \quad \theta = \frac{1}{n}-\frac{1}{N} = \frac{N-n}{nN}$$

Conditions under which these estimators are preferred

(A) When $\widehat{\rho}_{xy} > 0$

(i) $mse(\bar{y}_r) < mse(\bar{y})$ whenever $\widehat{\rho}_{xy} > \frac{cv(x)}{2cv(y)}$

(ii) $mse(\bar{y}_{ar1}) < mse(\bar{y})$ whenever $\widehat{\rho}_{xy} > \frac{cv(x)}{2cv(y)}(\frac{n}{N-n})$

(iii) $mse(\bar{y}_{ar2}) < mse(\bar{y})$ whenever $\widehat{\rho}_{xy} > \frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n})$

(iv) $mse(\bar{y}_{ar1}) < mse(\bar{y}_r)$ whenever

$$\frac{cv(x)}{2cv(y)}(\frac{n}{N-n}) < \widehat{\rho}_{xy} \le \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{n}{N-n})^2)}{(1-\frac{n}{N-n})}]$$

(v) $mse(\bar{y}_{ar2}) < mse(\bar{y}_r)$ whenever

$$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n}) < \widehat{\rho}_{xy} \le \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-\frac{N-n}{Nn-N+n})}]$$

(vi) $mse(\bar{y}_{ar2}) < mse(\bar{y}_{ar1})$ whenever

$$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n}) < \widehat{\rho}_{xy} \le \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-\frac{N-n}{Nn-N+n})}] \quad \text{and} \quad \frac{n}{N} \le \frac{N-n}{nN},$$

$(f \le \theta)$.

That is, our alternative ratio estimator, $\bar{y}_{ar2}$ will be preferred to all other estimators being considered here if:-

$$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n}) < \widehat{\rho}_{xy} \le \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-\frac{N-n}{Nn-N+n})}] \quad \text{and} \quad \frac{n}{N} \le \frac{N-n}{nN}.$$

(B) When $\widehat{\rho}_{xy} < 0$

(i) $mse(\bar{y}_P) < mse(\bar{y})$ whenever $\widehat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}$

(ii) $mse(\bar{y}_{aP1}) < mse(\bar{y})$ whenever $\widehat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}(\frac{n}{N-n})$

(iii) $mse(\bar{y}_{aP2}) < mse(\bar{y})$ whenever $\widehat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n})$

(iv) $mse(\bar{y}_{aP1}) < mse(\bar{y}_P)$ whenever

$$-\frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{n}{N-n})^2)}{(1-(\frac{n}{N-n}))}] \le \widehat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}(\frac{n}{N-n})$$

(v) $mse(\bar{y}_{aP2}) < mse(\bar{y}_P)$ whenever

$$-\frac{cv(x)}{2cv(y)}\left[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-(\frac{N-n}{Nn-N+n}))}\right] \leq \hat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}\left(\frac{N-n}{Nn-N+n}\right)$$

(vi) $mse(\bar{y}_{aP2}) < mse(\bar{y}_{aP1})$ whenever

$$-\frac{cv(x)}{2cv(y)}\left[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-\frac{N-n}{Nn-N+n})}\right] \leq \hat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}\left(\frac{N-n}{Nn-N+n}\right)$$

and $\frac{n}{N} \leq \frac{N-n}{nN}$

That is, our alternative product estimator, $\bar{y}_{ap2}$ will be preferred to other estimators being considered here if :-

$$-\frac{cv(x)}{2cv(y)}\left[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-(\frac{N-n}{Nn-N+n}))}\right] \leq \hat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}\left(\frac{N-n}{Nn-N+n}\right)$$

and $\frac{n}{N} \leq \frac{N-n}{nN}$

Illustration

To confirm these conditionality, we made use of life data on School enrolments in the 200 schools in Kwara State for 1984/85(X) and 1985/86(Y) and the samples are drawn with replacement and these drawn samples are:- 2,5,10,13,14,20,30,50,80 and 90.

RESULTS

Table 1. Showing the empirical results of all the estimators using life data on schools enrolment in 200 schools in kwara state during 1984/85(x) and 1985/86(y) calendar years

	Sample size 2	5	10	13	14
$cv(x)$	0.0232	0.4903	0.3961	0.5361	0.5225
$cv(y)$	0.0449	0.2442	0.2881	0.2849	0.2851
$\widehat{\rho}_{xy}$	1.0000	0.9903	0.7157	0.8012	0.7158
$\dfrac{cv(x)}{2cv(y)}$	0.2582	1.0036	0.6874	0.9410	0.9158
$\dfrac{cv(x)(n)}{2cv(y)(N-n)}$	0.0026	0.0257	0.0362	0.0654	0.0689
$\dfrac{cv(x)}{2cv(y)}(\dfrac{N-n}{Nn-N+n})$	0.2531	0.2431	0.0727	0.0729	0.0651
$\dfrac{cv(x)}{2cv(y)}[\dfrac{1-(\frac{n}{N-n})^2}{1-(\frac{n}{N-n})}]$	0.2608	1.0294	0.7236	1.0064	0.9841
$\dfrac{cv(x)}{2cv(y)}[\dfrac{1-(\frac{N-n}{Nn-N+n})^2}{1-(\frac{N-n}{Nn-N+n})}]$	0.5118	1.2467	0.7596	1.0139	0.9803
f	0.01	0.02	0.05	0.065	0.07
θ	0.495	0.195	0.095	0.072	0.066
$mse(\bar{y})$	3.96	72.9691	66.4535	59.4499	58.3057
$mse(\bar{y}_r)$	0.9261	76.8598	61.2922	90.7457	100.8757
$mse(\bar{y}_{ar1})$	3.9188	65.7238	59.9185	48.0042	47.9128
$mse(\bar{y}_{ar2})$	0.9657	19.9462	54.1098	46.8209	48.4236
$bias(\bar{y}_r)$	-0.0157	1.8806	0.6561	1.1983	1.1462
$bias(\bar{y}_{ar1})$	-0.0003	-0.0469	-0.0375	-0.0617	-0.0554
$bias(\bar{y}_{ar2})$	-0.0318	-0.4436	-0.0748	-0.0688	-0.0524

	Sample size 20	30	50	80	90
$cv(x)$	0.4696	0.4772	0.4413	0.4188	0.4172
$cv(y)$	0.2669	0.2470	0.2714	0.2967	0.3109
$\hat{\rho}_{xy}$	0.7679	0.7584	0.5735	0.6422	0.6272
$\frac{cv(x)}{2cv(y)}$	0.8796	0.9659	0.8131	0.7057	0.6708
$\frac{cv(x)(n)}{2cv(y)(N-n)}$	0.0977	0.1705	0.2710	0.4705	0.5488
$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n})$	0.0414	0.0282	0.0124	0.0053	0.0041
$\frac{cv(x)}{2cv(y)}[\frac{1-(\frac{n}{N-n})^2}{1-(\frac{n}{N-n})}]$	0.9774	1.1364	1.0841	1.1762	1.2196
$\frac{cv(x)}{2cv(y)}[\frac{1-(\frac{N-n}{Nn-N+n})^2}{1-(\frac{N-n}{Nn-N+n})}]$	0.9211	0.9941	0.8255	0.7110	0.6749
f	0.1	0.15	0.25	0.40	0.45
θ	0.045	0.028	0.015	0.008	0.006
$mse(\bar{y})$	42.9578	28.9444	26.8665	23.8829	24.0653
$mse(\bar{y}_r)$	59.8424	52.1548	47.8035	28.1647	26.8780
$mse(\bar{y}_{ar1})$	31.7022	17.3422	18.0576	16.1651	19.9230
$mse(\bar{y}_{ar2})$	37.7836	26.5631	26.1199	23.5585	23.8179
$bias(\bar{y}_r)$	0.6473	0.5071	0.2949	0.1363	0.1143
$bias(\bar{y}_{ar1})$	-0.0557	-0.0578	-0.0535	-0.0759	-0.0211
$bias(\bar{y}_{ar2})$	-0.0236	-0.0096	-0.0024	-0.0009	-0.0006

DISCUSSION

So, from this table, we observed that

(a) when n=5, 10 and 13, $mse(\bar{y}_{ar2}) < mse(\bar{y}_{ar1}) < mse(\bar{y}_r) < mse(\bar{y})$ indicating that (i)

$$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n}) < \hat{\rho}_{xy} < \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2}{(1-(\frac{N-n}{Nn-N+n}))}]$$

and (ii) $f \le \theta$, that is $\frac{n}{N} \le \frac{N-n}{nN}$ meaning that our alternative ratio estimator $\bar{y}_{ar2}$ is preferred to the other estimators being considered but when n=2, $mse(\bar{y}_r) < mse(\bar{y}_{ar2}) < mse(\bar{y}_{ar1}) < mse(\bar{y})$ still $f < \theta$ but

$$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n}) < \hat{\rho}_{xy} > \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2}{(1-(\frac{N-n}{Nn-N+n}))}]$$ which means that $\bar{y}_{ar2}$ is not preferred. Again when n=14,20,30,50,80 and 90, although

$$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n}) < \hat{\rho}_{xy} < \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2}{(1-(\frac{N-n}{Nn-N+n}))}]$$

but $f > \theta$ that is $\frac{n}{N} > \frac{N-n}{nN}$ meaning that our alternative ratio estimator, $\bar{y}_{ar2}$ is not preferred.

That is when the sample size (n) is :

(i) 2, 0.2531 < 1 > 0.5118 and 0.01 < 0.495, ($f < \theta$) ,hence $\bar{y}_{ar2}$ is not preferred.

(ii) 5, 0.2431 < 0.9903 < 1.2467 and 0.02 < 0.195, $\bar{y}_{ar2}$ is preferred

(iii) 10, 0.0727 < 0.7157 < 0.7596 and 0.05 < 0.095, $\bar{y}_{ar2}$ is preferred

(iv) 13, 0.0729 < 0.8012 < 1.0139 and 0.065 < 0.072, $\bar{y}_{ar2}$ is preferred

(v) 14, 0.0651 < 0.7158 < 0.9803 and 0.07 > 0.066, $\bar{y}_{ar2}$ is not preferred

(vi) 20, 0.0414 < 0.7679 < 0.9211 and 0.1 > 0.045, $\bar{y}_{ar2}$ is not preferred

(vii) 30, 0.0282 < 0.7584 < 0.9941 and 0.15 > 0.028, $\bar{y}_{ar2}$ is not preferred

(viii) 50, 0.0124 < 0.5735 < 0.8255 and 0.25 > 0.015, $\bar{y}_{ar2}$ is not preferred

(ix) 80, 0.0053 < 0..6422 < 0.7110 and 0.40 > 0.008, $\bar{y}_{ar2}$ is not preferred

(x) 90, 0.0041 < 0.6272 < 0.6749 and 0.45 > 0.006, $\bar{y}_{ar2}$ is not preferred

CONCLUSION

We have seen from the empirical result shown above that our alternative ratio and product estimators , $\bar{y}_{ar2}$ and $\bar{y}_{ap2}$ would be preferred to all other estimators being considered here whenever:-

$$\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n}) < \hat{\rho}_{xy} \le \frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-(\frac{N-n}{Nn-N+n}))}]$$ and

$$-\frac{cv(x)}{2cv(y)}[\frac{(1-(\frac{N-n}{Nn-N+n})^2)}{(1-(\frac{N-n}{Nn-N+n}))}] \le \hat{\rho}_{xy} < -\frac{cv(x)}{2cv(y)}(\frac{N-n}{Nn-N+n})$$

respectively and in addition, $f \le \theta$, that is, $\frac{n}{N} \le \frac{N-n}{nN}$. However, these two conditions must be fully satisfied before any meaningful conclusion could be drawn.

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